A STUDY OF ANALYTIC SURFACES BY MEANS OF A PROJECTIVE THEORY OF ENVELOPES(1)

BY

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1. **Introduction.** Let Q_x denote a quadric associated with a generic point x of an analytic surface S, and let C_λ denote a curve of S passing through x. Let X denote a point of C_λ in the neighborhood of x. The limit of the curve of intersection of the quadrics Q_x and Q_X as X approaches x is called the *characteristic curve of* Q_x *with respect to the curve* C_λ *at* x. The quadric cone which contains this characteristic curve and has its vertex at x will be called the *characteristic cone* $\mathcal{C}_\lambda$ *of* Q_x *at* x.

This paper applies a projective theory of envelopes [2](2) to determine and study the cones $\mathcal{C}_\lambda$ associated with significant families of quadrics having contact of the second order with S at x.

The geometric inter-relations of the cone $\mathcal{C}_\lambda$, the quadric Q_x, and the curve C_λ are investigated. The principal results are geometric characterizations of significant families of quadrics, systems of hypergeodesics, the canonical pencil, projective curvatures of a curve C_λ, the general transformation of Čech, the curves of Darboux and Segre, and a class of generalized pangeodesics. Among the quadrics characterized are the Moutard pencil of quadrics, the Davis quadrics, and the asymptotic osculating quadrics of Bompiani. These quadrics have been defined, heretofore, by distinct and apparently unrelated properties. They are here shown to be members of a system of quadrics characterized by a special property of their characteristic cones. Other properties serve to characterize an invariant pencil of Darboux quadrics.

2. **Preliminaries.** Let an analytic non-ruled surface S be referred to its asymptotic net as parametric, and let the projective homogeneous coordinates of a general point x of S be normalized so that they satisfy the Fubini canonical system of differential equations

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v, \qquad \theta = \log \beta\gamma. \end{aligned} \tag{2.1}$$

Since the points x, x_u, x_v, x_{uv} are linearly independent, the general homogeneous coordinates of any point X may be written in the form

$$X = x^0 x + x^1 x_u + x^2 x_v + x^3 x_{uv}.$$

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(2) Numbers in brackets refer to the references cited at the end of the paper.

The local coordinates of X with respect to the tetrahedron (x, x_u, x_v, x_{uv}) are proportional to x^0, x^1, x^2, x^3 if the unit point of the reference frame is suitably chosen. Referred to this coordinate system the three-parameter family of quadric surfaces having contact of the second order with S at x has the equation

$$x^1x^2 - x^0x^3 + k_1x^1x^3 + k_2x^2x^3 + k_3(x^3)^2 = 0, \tag{2.2}$$

where k_1, k_2, and k_3 are arbitrary functions of u, v. If, and only if, $k_1 = k_2 = 0$, the quadrics (2.2) are quadrics of Darboux defined by

$$x^1x^2 - x^0x^3 + k_3(x^3)^2 = 0. \tag{2.3}$$

3. **The characteristic cone. Conjugate and triple point tangents.** A projective theory of envelopes is based on the use of the formulas for differentiation of local point coordinates. These formulas are [2, pp. 27–28]

$$\partial x^i/\partial u^\alpha = -\overset{h}{x}\overset{i}{\Gamma}_{h\alpha}, \qquad i, h = 0, 1, 2, 3, \quad \alpha = 1, 2, \quad \overset{1}{u} = u, \quad \overset{2}{u} = v, \tag{3.1}$$

in which the repeated index h denotes summation through the indicated range, and the functions $\Gamma^i_{h\alpha}$ are (in the case of the present coordinate system) related to the coefficients of (2.1) by the equations

$$\begin{gathered}
\overset{i}{\Gamma}_{0\alpha} = \overset{i}{\delta}_\alpha, \qquad \overset{i}{\Gamma}_{\alpha\epsilon} = \overset{i}{\Gamma}_{\epsilon\alpha} = \overset{i}{\delta}_3, \qquad \alpha \neq \epsilon, \qquad \epsilon, \alpha = 1, 2, \\
\overset{0}{\Gamma}_{11} = p, \qquad \overset{\alpha}{\Gamma}_{\alpha\alpha} = \overset{3}{\Gamma}_{3\alpha} = \frac{\partial\theta}{\partial u^\alpha}, \qquad \overset{3}{\Gamma}_{\alpha\alpha} = 0, \qquad \overset{0}{\Gamma}_{22} = q, \\
\overset{1}{\Gamma}_{22} = \gamma, \qquad \overset{2}{\Gamma}_{11} = \beta, \qquad \overset{\alpha}{\Gamma}_{3\alpha} = \kappa, \qquad \overset{1}{\Gamma}_{32} = \chi, \\
\overset{2}{\Gamma}_{31} = \pi, \qquad \overset{0}{\Gamma}_{31} = p_v + \beta q, \qquad \overset{0}{\Gamma}_{32} = q_u + \gamma p, \\
\kappa = \beta\gamma + \theta_{uv}, \qquad \pi = p + \beta_v + \beta\theta_v, \qquad \chi = q + \gamma_u + \gamma\theta_u,
\end{gathered}$$

in which the deltas denote the Kronecker symbols.

Let C_λ denote a curve of S defined by the curvilinear differential equation $dv - \lambda(u, v)du = 0$. Let equation (2.2) be represented by $f = 0$. The equations of the characteristic curve of (2.2) are therefore given by

$$f = 0, \qquad f_u + \lambda f_v = 0, \tag{3.2}$$

in which the right members of (3.1) are substituted for the partial derivatives of local point coordinates. The second equation of (3.2) is found to be

$$\begin{aligned}
&(\beta + k_1\lambda)(x^1)^2 + (\theta_u + \lambda\theta_v + k_1 + k_2\lambda)x^1x^2 + (\gamma\lambda + k_2)(x^2)^2 \\
&\quad + (k_1 + k_2\lambda - \theta_u - \lambda\theta_v)x^0x^3 + (\pi + \kappa\lambda + k_1\theta_u + k_1\lambda\theta_v + k_1\theta_u \\
&\quad + k_2\beta - p - k_1' + 2k_3\lambda)x^1x^3 + (\kappa + \chi\lambda + k_2\theta_u + k_2\lambda\theta_v \\
&\quad + k_1\gamma\lambda + k_2\theta_v\lambda - q\lambda - k_2' + 2k_3)x^2x^3 + (k_1\kappa + k_1\lambda\chi + k_2\pi \\
&\quad + k_2\lambda\kappa - p_v - \beta q - q_u\lambda - \gamma p\lambda - k_3' + 2k_3\theta_u + 2k_3\lambda\theta_v)(x^3)^2 = 0,
\end{aligned} \tag{3.3}$$

in which accents indicate differentiation with respect to u.

The equation of the characteristic cone $\mathcal{C}_\lambda$ is found, by eliminating x^0x^3 between equations (2.2) and (3.3), to be

$$\begin{aligned}(3.4)\qquad &(\beta + k_1\lambda)(x^1)^2 + 2(k_1 + k_2\lambda)x^1x^2 + (\gamma\lambda + k_2)(x^2)^2\\ &+ (k_1^2 + k_1k_2\lambda + \theta_u k_1 + \beta k_2 - k_1' + \beta\psi + \kappa\lambda + 2k_3\lambda)x^1x^3\\ &+ (k_2^2\lambda + k_1k_2 + k_2\theta_v\lambda + k_1\gamma\lambda - k_2' + \gamma\phi\lambda + \kappa + 2k_3)x^2x^3\\ &+ [(k_1 + k_2\lambda)(\kappa + k_3) + \pi k_2 + \chi k_1\lambda - p_v - \beta q - q_u\lambda - \gamma p\lambda\\ &+ \theta' k_3 - k_3'](x^3)^2 = 0,\end{aligned}$$

where $\psi = (\log \beta^2\gamma)_v$, $\phi = (\log \beta\gamma^2)_u$.

Let t_τ denote the tangent line to S at x whose direction is τ. Since the vertex of the cone $\mathcal{C}_\lambda$ is the point x, the polar plane of a point P of t_τ with respect to the cone $\mathcal{C}_\lambda$ is independent of the choice of P on t_τ. The equation of this polar plane is

$$\begin{aligned}(3.5)\qquad &2[\beta + (\lambda + \tau)k_1 + \lambda\tau k_2]x^1 + 2[\gamma\lambda\tau + k_1 + (\lambda + \tau)k_2]x^2\\ &+ [(k_1^2 + \theta_u k_1 + \beta k_2 - k_1' + \beta\psi) + (\lambda + \tau)(k_1k_2 + \kappa + 2k_3)\\ &+ (\lambda k_2^2 + \theta_v\lambda k_2 + \gamma\lambda k_1 - k_2' + \gamma\lambda\phi)\tau]x^3 = 0.\end{aligned}$$

Let π_x denote the tangent plane of S at x. The intersection of (3.5) and π_x is in the direction defined by

$$(3.6)\qquad x^2/x^1 = -[\beta + (\lambda + \tau)k_1 + \lambda\tau k_2]/[\gamma\lambda\tau + (\lambda + \tau)k_2 + k_1].$$

The equations of the lines of intersection of the cone $\mathcal{C}_\lambda$ with π_x are

$$(3.7)\qquad x^3 = 0,\ (\beta + k_1\lambda)(x^1)^2 + 2(k_1 + k_2\lambda)x^1x^2 + (\gamma\lambda + k_2)(x^2)^2 = 0.$$

The directions of these lines are conjugate directions $\pm\eta$ if, and only if, $k_1+\lambda k_2=0$, where

$$\eta = [-(\beta + k_1\lambda)/(\gamma\lambda + k_2)]^{1/2}.$$

It follows that if $k_1=k_2=0$,

$$\lambda = -\beta/\gamma\eta^2.$$

The characteristic cone $\mathcal{C}_\lambda$ determined by a quadric of Darboux will be denoted by $\mathcal{C}_{\lambda,D}$.

Theorem 3.1. *The directions of the lines of intersection of a characteristic cone $\mathcal{C}_{\lambda,D}$ with the plane π_x are conjugate directions $\pm\eta$. The direction λ is the R_η-correspondent of η* [1, p. 392].

If λ coincides with one of the directions of the lines (3.7), λ is found to be a solution of the equation

$$(3.8)\qquad \beta + 3k_1\lambda + 3k_2\lambda^2 + \gamma\lambda^3 = 0$$

for the directions of the triple point tangents of the curve of intersection of S and the quadric (2.2).

THEOREM 3.2. *The cone* $\mathcal{C}_\lambda$ *intersects* π_x *in the tangent to the curve* $\mathcal{C}_\lambda$ *if, and only if,* t_λ *is one of the triple point tangents of the curve of intersection of the surface* S *with the quadric* (2.2).

Let N_λ denote the conjugate net of S whose tangents at x are in the directions $\pm\lambda$. On making use of the condition, $k_1+\lambda k_2=0$, that the cone $\mathcal{C}_\lambda$ intersect π_x in conjugate tangents together with the condition (3.8), the following theorem results.

THEOREM 3.3. *The cone* $\mathcal{C}_\lambda$ *intersects the plane* π_x *in the tangents at* x *to the conjugate net* N_λ *if, and only if,* λ *is a direction of Darboux.*

4. **Condition that $\mathcal{C}_{\lambda,D}$ reduce to two planes. Quadric of Lie.** The condition that $\mathcal{C}_{\lambda,D}$ reduce to two planes is readily found by equating to zero k_1, k_2, and the discriminant of (3.4) to be

$$\begin{aligned}\beta^{-1}(2k_3+\kappa)^2\lambda^3 &+ (4k_{3_v} - 4\theta_v k_3 + 4q_u + 4\gamma p + 2\psi\kappa \\ &+ 4\psi k_3 + \gamma\phi^2)\lambda^2 + (4k_{3_u} - 4\theta_u k_3 + 4p_v + 4\beta q + 2\phi\kappa \\ &+ 4\phi k_3 + \beta\psi^2)\lambda + \gamma^{-1}(2k_3+\kappa)^2 = 0.\end{aligned}$$

THEOREM 4.1. *The cone* $\mathcal{C}_{\lambda,D}$ *reduces to two intersecting planes for each of three directions. Two of these directions are the asymptotic directions if, and only if, the quadric is the quadric of Lie* ($k_3=-\kappa/2$)*; the third direction is then defined by*

$$\lambda = -\frac{8p_v + 2(\beta\psi)_v - 2\beta\gamma\phi + 4\beta q + \beta\psi^2}{8q_u + 2(\gamma\phi)_u - 2\beta\gamma\psi + 4\gamma p + \gamma\phi^2}\,.$$

5. **Polar planes with respect to $\mathcal{C}_{\lambda,D}$.** Let $p_{\mu,\lambda}$ denote the polar plane of an arbitrary point on t_μ with respect to the cone $\mathcal{C}_{\lambda,D}$. The equation of the plane $p_{\mu,\lambda}$ is found to be

$$(5.1)\qquad 2\beta x^1 + 2\gamma\lambda\mu x^2 + [\beta\psi + \gamma\phi\lambda\mu + (\lambda+\mu)(\kappa+2k_3)]x^3 = 0.$$

This plane intersects π_x in a line whose direction coincides with λ if μ is defined by

$$(5.2)\qquad \mu = -\beta/\gamma\lambda^2.$$

THEOREM 5.1. *The polar plane* $p_{\mu,\lambda}$ *passes through the tangent* t_λ *of* S *at* x *if and only if* t_μ *is the* R_λ*-correspondent of* t_λ*. The polar plane* $p_{\lambda,\lambda}$ *passes through the* R_λ*-correspondent of* t_λ*.*

The equation of an arbitrary plane through t_λ is of the form

$$(5.3)\qquad 2\lambda^2 x^1 - 2\lambda x^2 + \rho x^3 = 0.$$

The equation of the polar plane $p_{h\lambda,\lambda}$, $h=\text{const.}$, may be found at once from (5.1) to be

$$\beta(2x^1 + \psi x^3) + (1 + h)(\kappa + 2k_3)\lambda x^3 + \gamma(2x^2 + \phi x^3)h\lambda^2 = 0.$$

If $(1+h)(\kappa+2k_3)=0$, this plane generates a pencil as λ varies, whose axis is defined by the equations

$$2x^1 + \psi x^3 = 0, \qquad 2x^2 + \phi x^3 = 0.$$

THEOREM 5.2. *The polar plane $p_{-\lambda,\lambda}$ is independent of the choice of the quadric of Darboux and for each value of λ passes through the directrix d' of Wilczynski. The polar plane $p_{h\lambda,\lambda}$, $h\neq -1$, for each direction λ passes through the directrix of Wilczynski if, and only if, the quadric of Darboux is the quadric of Lie.*

To find the equations of the planes through t_λ which are tangent to the cone $\mathcal{C}_{\lambda,D}$, equate k_1 and k_2 to zero in (3.4), eliminate x^3 between the resulting equations (5.3) and (3.4), and set the discriminant of this eliminant equal to zero. This discriminant equation is found to be of the form

$$\beta\gamma\rho^2 + 2[\beta\gamma\lambda(\phi - \psi\lambda) + (\beta - \gamma\lambda^3)(\kappa + 2k_3)]\rho + (1) = 0, \tag{5.4}$$

in which the terms denoted by (1) are immaterial in the present considerations. The roots of this quadratic equation in ρ are the values of ρ for which the plane (5.3) is tangent to $\mathcal{C}_{\lambda,D}$. The harmonic conjugate of π_x with respect to the two tangent planes to $\mathcal{C}_{\lambda,D}$ through t_λ, therefore, has the equation (5.3) where ρ is one-half the sum of the roots of (5.4), that is

$$\rho = \psi\lambda^2 - \phi\lambda + (\gamma\lambda^3 - \beta)(\kappa + 2k_3)/\beta\gamma. \tag{5.5}$$

This plane is identical with the polar plane $p_{\mu,\lambda}$ where μ is defined by (5.2). The following theorem incorporates the result just proved with others which may be readily derived.

THEOREM 5.3. *The polar plane $p_{\mu,\lambda}$ ($p_{\lambda,\lambda}$), where t_μ is the R_λ-correspondent of t_λ, is the harmonic conjugate of π_x with respect to the two tangent planes to $\mathcal{C}_{\lambda,D}$ through t_λ (t_μ). The plane $p_{\mu,\lambda}$, for each direction λ, passes through the directrix d' of Wilczynski if, and only if, the quadric of Darboux is the quadric of Lie.*

The osculating plane of the curve C_λ at x has for its equation (5.3) where ρ is defined by

$$\rho = \lambda' + \beta - \theta_u\lambda + \theta_v\lambda^2 - \gamma\lambda^3. \tag{5.6}$$

This plane coincides with the plane $p_{\mu,\lambda}$ where μ is defined by (5.2) if, and only if, the values for ρ defined by (5.5) and (5.6) are identical. This condition demands that C_λ be a curve of a family of hypergeodesics defined by the equation

$$\lambda' = -\eta\beta - (\log\gamma)_u\lambda + (\log\beta)_v\lambda^2 + \eta\gamma\lambda^3, \tag{5.7}$$

where $\eta=(2\beta\gamma+\theta_{uv}+2k_3)/\beta\gamma$. A distinct family of hypergeodesics is thus associated with each quadric of Darboux. *The cusp-axis of any family of these hypergodesics is the directrix d' of Wilczynski.*

Because of the relationships of these hypergeodesics to the quadrics of Darboux and the directrix of Wilczynski it seems appropriate to call them the *W_D-geodesics.*

THEOREM 5.4. *The polar plane $p_{\mu,\lambda}$ where t_μ is the R_λ-correspondent of t_λ coincides with the osculating plane of C_λ at x if, and only if, the curve C_λ is a W_D-geodesic defined by* (5.7). *If η has the value* 1, *the W_D-geodesics are the union curves of the congruence Γ' of the directrices of Wilczynski.*

Another system of W_D-geodesics may be characterized in association with each quadric of Darboux by replacing in the above characterization the plane $p_{\mu,\lambda}$ by the plane $p_{-\mu,-\lambda}$. The equation of this system is again of the form (5.7), but with η defined by the relation $\eta=-(\theta_{uv}+2k_3)/\beta\gamma$. *This system of W_D-geodesics defined with respect to a quadric can coincide with the first system defined for the same quadric if, and only if, the quadric is the quadric of Lie. In this case the W_D-geodesics are the union curves of the congruence Γ' of directrices of Wilzcynski. If $\eta=-1$, the second system of W_D-geodesics consists of the dual union curves of the congruence Γ of directrices of Wilczynski.*

6. **W_D-geodesic curvature and the Darboux pencil.** Let π_0 denote the osculating plane of the curve C_λ at x, let π_1 denote the plane containing the tangent t_λ and the cusp-axis [5, p. 190] of the pencil of conjugate nets defined by

$$dv^2-\lambda^2h^2du^2=0, \qquad h=\text{const.},$$

and let π_2 denote the plane $p_{\mu,\lambda}$ where μ is defined by (5.2). The equations of π_0, π_2, and π_1 are of the form (5.3) where the corresponding functions ρ are given by (5.6), (5.5), and $\rho=\lambda'-\theta_u\lambda+\theta_v\lambda^2$, respectively. The W_D-geodesic curvature of the curve C_λ at x will be defined as the cross ratio of the planes π_x, π_0, π_1, π_2 and will be denoted by σ_1, which is found to be given by

$$\begin{aligned}\sigma_1 &= (\pi_x, \pi_0, \pi_1, \pi_2)\\ &= [\lambda'+\gamma^{-1}\eta+(\log\gamma)_u\lambda-(\log\beta)_v\lambda^2-\beta^{-1}\eta\lambda^3]/(\beta-\gamma\lambda^3)\end{aligned}$$

where $\eta=\theta_{uv}+2\beta\gamma+2k_3$. Hence

$$\begin{aligned}\lambda' = &-\gamma^{-1}[\theta_{uv}+(2-\sigma_1)\beta\gamma+2k_3]-(\log\gamma)_u\lambda\\ &+(\log\beta)_v\lambda^2+\beta^{-1}[\theta_{uv}+(2-\sigma_1)\beta\gamma+2k_3]\lambda^3.\end{aligned} \tag{6.1}$$

This equation defines a system of hypergeodesics. A comparison of the equations (5.7) and (6.1) reveals that *a curve whose W_D-geodesic curvature vanishes identically is a W_D-geodesic of the system* (5.7).

The hypergeodesics (6.1) are the union curves of the congruence of the directrices of Wilczynski if, and only if,

$$k_3 = -[(1-\sigma_1)\beta\gamma + \theta_{uv}]/2. \tag{6.2}$$

This condition yields the result that *any quadric of Darboux can be characterized by assigning a W_D-geodesic curvature $\sigma_1(u, v)$ to all of the union curves of the congruence of the directrices of Wilczynski.* In particular if a constant h is the value assigned to σ_1, the equation of the resulting quadric of Darboux is

$$2(x^1x^2 - x^0x^3) - [(1-h)\beta\gamma + \theta_{uv}](x^3)^2 = 0. \tag{6.3}$$

This is the equation of a well known invariant pencil of Darboux quadrics. The form of equation (6.1) is such that the following theorem may be easily proved.

THEOREM 6.1. *The W_D-geodesic curvature of a union curve of a congruence Γ' at x is independent of the direction of the curve at x if, and only if, Γ' is the congruence of the directrices d' of Wilczynski and k_3 is defined by* (6.2). *This curvature is constant along the curve C_λ if the quadric is a member of the Darboux invariant pencil* (6.3). *The values* 0, 1, *and* 1/3 *for σ_1 correspond to the quadrics of Lie, Wilczynski, and Fubini, respectively.*

The form of the right member of the equation which defines σ_1 reveals that *the W_D-geodesic curvature of the curves of Segre, independent of the choice of the quadric of Darboux, is undefined.*

7. **Pangeodesic curvature and the Darboux pencil.** The characteristic curve, of the quadric (2.2), which corresponds to the direction λ at x includes both asymptotic tangents at x if, and only if, the coefficients of $(x^1)^2$ and $(x^2)^2$ in equation (3.3) vanish; that is, equation (2.2) becomes

$$x^1x^2 - x^0x^3 - \beta\lambda^{-1}x^1x^3 - \gamma\lambda x^2x^3 + k_3(x^3)^2 = 0. \tag{7.1}$$

For the quadric (7.1) equation (3.6) reduces to

$$x^2/x^1 = -\tau.$$

This is the direction conjugate to τ.

Let $\pi_{-\lambda}$ denote the polar plane of an arbitrary point on $t_{-\lambda}$ with respect to the characteristic cone $\mathcal{C}_\lambda$ of (7.1). The equation of this plane is (5.3) where

$$\begin{aligned} \rho = &-\lambda' + (\beta+\gamma\lambda^3)^{-1}[\beta^2 - \beta(\log\gamma)_u\lambda + (3\beta_v + \beta(\log\gamma)_v)\lambda^2 \\ &- (3\gamma_u + \gamma(\log\beta)_u)\lambda^4 + \gamma(\log\beta)_v\lambda^5 - \gamma^2\lambda^6]. \end{aligned} \tag{7.2}$$

This plane intersects π_x in the tangent t_λ. It coincides with the osculating plane of C_λ if, and only if, the value of ρ in (7.2) is equal to that defined by (5.6). This condition reduces to the differential equation which defines the system of pangeodesics

$$2(\beta+\gamma\lambda^3)\lambda' = \beta_u\lambda + 2\beta_v\lambda^2 - 2\gamma_u\lambda^4 - \gamma_v\lambda^5.$$

THEOREM 7.1. *The plane $\pi_{-\lambda}$ coincides with the osculating plane of C_λ at x*

if, and only if, the curve C_λ is a pangeodesic.

The pangeodesic curvature of a curve C_λ will be defined as the cross ratio of $\pi_{-\lambda}$ with respect to the fundamental set of planes π_x, π_0, π_1 (considered in §6), and denoted by $\sigma_2(u, v)$. The formula for σ_2 is found to be

$$\begin{aligned}\sigma_2 &= (\pi_x, \pi_0, \pi_1, \pi_{-\lambda})\\ &= [-2(\beta + \gamma\lambda^3)\lambda' + \beta_u\lambda + 2\beta_v\lambda^2 - 2\gamma_u\lambda^4 - \gamma_v\lambda^5]/(\gamma^2\lambda^6 - \beta^2).\end{aligned}$$

THEOREM 7.2. *The pangeodesic curvature of a curve C_λ with respect to a quadric* (7.1) *is independent of the particular choice of the quadric. The pangeodesic curvature of a pangeodesic is equal to zero.*

Lane [5, p. 148] has shown that the asymptotic osculating quadric Q_v of Bompiani becomes a quadric of Darboux when C_λ is tangent to the asymptotic u-curve at x, and the quadric of Darboux is given by (2.3) where

$$k_3 = -(\gamma\lambda' + \beta\gamma + \theta_{uv})/2. \tag{7.3}$$

If C_λ is tangent to an asymptotic curve at x, say for example, the u-curve, the pangeodesic curvature σ_2 of the curve C_λ is defined by

$$\sigma_2 = 2\lambda'/\beta.$$

Then

$$\lambda' = \beta\sigma_2/2. \tag{7.4}$$

Now if λ' in formula (7.3) is replaced by its value in (7.4), the following theorem results.

THEOREM 7.3. *When C_λ is tangent to the asymptotic u-curve at x and has a constant pangeodesic curvature, the osculating quadric Q_v of Bompiani becomes the member of the Darboux pencil* (6.3) *for which $h = -\sigma_2/2$.*

COROLLARY 7.4. *If C_λ is tangent at x to the asymptotic u-curve, the osculating quadratic Q_v of Bompiani becomes the quadric of Lie, the quadric of Wilczynski, or the quadric of Fubini, according as the pangeodesic curvature of C_λ is* 0, -2, *or* $-2/3$, *respectively.*

The above characterization of the quadric of Lie is equivalent to the usual definition since the curve C_λ, above defined, becomes the asymptotic u-curve in case its pangeodesic curvature is equal to zero.

Similar results to those above can be obtained by using C_λ in the direction of the asymptotic v-curve and the asymptotic osculating quadric Q_u.

8. **The canonical pencil in the canonical plane.** Let l' denote the line passing through x and the point z whose general coordinates are given by

$$z = x_{uv} - ax_u - bx_v.$$

The plane (5.3) passes through l' if, and only if,

$$\rho = 2a\lambda^2 - 2b\lambda.$$

The plane $p_{\mu,\lambda}$ with μ defined by (5.2) contains l' when the quadric is chosen such that

$$k_3 = -\kappa/2 + \omega,$$

where

$$\omega = \beta\gamma[(2a - \psi)\lambda^2 - (2b - \phi)\lambda]/2(\gamma\lambda^3 - \beta).$$

This quadric of Darboux intersects l' in the point P_1 whose general coordinates are given by

$$z + (ab - \kappa/2 + \omega)x.$$

Let P_h, h=const., denote the point whose general coordinates are given by

$$z + (ab - \kappa/2 + h\omega)x.$$

Therefore, the point P_0 is the intersection other than x of the quadric of Lie with the line l', and the point P_∞ is the point x. Hence, the point P_h is geometrically determined by the cross ratio

$$(P_\infty, P_0, P_1, P_h) = h.$$

Let D_h denote the quadric of Darboux, which passes through the point P_h. The equation of D_h is (2.3) where k_3 is given by

$$k_3 = h\omega - \kappa/2.$$

Let ϑ_h denote the polar plane $p_{\mu,\lambda}$ where μ is given by (5.2) and the quadric used in the definition is the quadric D_h. The equation of the plane ϑ_h is (5.3) where

$$\rho = [2ah + (1 - h)\psi]\lambda^2 - [2bh + (1 - h)\phi]\lambda. \tag{8.1}$$

From (8.1) and the form of the coordinates of P_h it follows that *the correspondence between the points P_h and the planes ϑ_h is a projectivity.*

As λ varies and h is held fixed, the plane ϑ_h generates a pencil whose axis is the line defined by $x^1 + a'x^3 = 0$, $x^2 + b'x^3 = 0$, where

$$a' = [2ah + (1 - h)\psi]/2, \qquad b' = [2bh + (1 - h)\phi]/2.$$

THEOREM 8.1. *When l' is the canonical line $c'(k)$, for which $a = -k\psi$, $b = -k\phi$, the axis of the pencil of planes ϑ_h is the canonical line $c'(k')$ where $k' = (1 - h - 2kh)/2$. If l' is the projective normal, $k = 0$ and $k' = (1 - h)/2$.*

When l' is the R'_λ-associate [1, p. 391] of the canonical line $c'(k)$, $a = -k\psi$

$-\gamma\lambda$ and $b=-k\phi-\beta/\lambda$, the equation of the plane ϑ_h is (5.3) where

$$\rho = 2h\beta + (2kh + h - 1)\phi\lambda - (2kh + h - 1)\psi\lambda^2 - 2h\gamma\lambda^3. \tag{8.2}$$

THEOREM 8.2. *The plane ϑ_h with ρ defined by* (8.2) *coincides with the osculatlating plane of the curve C_λ if, and only if, C_λ is a curve in the system of hypergeodesics defined by*

$$\begin{aligned}\lambda' = (2h - 1)\beta + [\theta_u + (2kh + h - 1)\phi]\lambda \\ - [\theta_v + (2kh + h - 1)\psi]\lambda^2 - (2h - 1)\gamma\lambda^3.\end{aligned} \tag{8.3}$$

The cusp-axis of this system is the canonical line which is the axis of the pencil of planes ϑ_h with ρ given by (8.1).

The above constructions based on any given canonical line $c'(k)$ yield any desired canonical line $c'(k')$ in the canonical plane for a suitable choice of D_h. The constructions for $h=0$ and $h=\infty$ yield, independently of k, the directrix d' of Wilczynski and the canonical tangent t', respectively.

9. **The K-associate quadrics.** Quadrics having contact of second order with S at x exist which are such that their characteristic cones for an arbitrary direction λ are tangent to π_x. There exists a two-parameter family of those quadrics which are characterized by the additional property that at every point x of S the direction μ of the line of contact of the characteristic cones of the family with π_x makes a constant cross ratio with the fundamental set which consists of the asymptotic directions and the direction λ at x. That is, $(\infty, 0, \lambda, \mu)=K$, $K=$const. From this cross ratio equation it is clear that

$$\mu = K\lambda.$$

The quadrics thus defined will be called the *K-associate quadrics of S at x* and such a quadric will be denoted by Q_K. A characteristic cone of a quadric Q_K with respect to the curve C_λ will be denoted by $\mathcal{C}_{\lambda,K}$.

To verify the above assertion and to obtain the equation of the K-associate quadrics of S note from equations (3.7) that the lines of intersection of $\mathcal{C}_\lambda$ with π_x coincide if, and only if,

$$(k_1 + \lambda k_2)^2 = (\gamma\lambda + k_2)(\beta + \lambda k_1). \tag{9.1}$$

The direction of the line of contact is given by

$$x^2/x^1 = -(k_1 + \lambda k_2)/(\gamma\lambda + k_2). \tag{9.2}$$

Demanding that this direction coincide with $K\lambda$ and solving (9.1) and (9.2) for k_1 and k_2 yields

$$\begin{aligned}k_1 &= [K^2\gamma\lambda^3 - (K + 1)\beta]/(K^2 + K + 1)\lambda, \\ k_2 &= [\beta - (K^2 + K)\gamma\lambda^3]/(K^2 + K + 1)\lambda^2.\end{aligned} \tag{9.3}$$

Since equation (2.2) subject to conditions (9.3) involves two arbitrary parameters K, k_3, the K-associate quadrics form a two-parameter family.

The quadrics of the *Moutard pencil* and the *quadrics of Davis* for a direction λ form the one-parameter families of K-associate quadrics for which $K=1$, and $K=-1$, respectively. The one-parameter families of K-associate quadrics for which $K=0$ and $K=\infty$ contain the asymptotic osculating quadrics Q_u and Q_v of Bompiani for the curve C_λ, respectively. It seems appropriate, therefore, to call these families for which $K=0$ and $K=\infty$ the families of *asymptotic associate quadrics*.

10. Transformation of Čech and conjugate directions. The transformation Σ_h of Čech is a one-to-one correspondence between an arbitrary point P in π_x and a plane π which passes through the point x. The purpose of this section is to present new geometric characterizations of this transformation and of conjugate directions.

Let y^0, y^1, y^2, 0 and 0, ξ_1, ξ_2, ξ_3 denote local homogeneous coordinates of the point P and plane π with respect to the reference tetrahedron whose vertices are x, x_u, x_v, x_{uv}. The point P and plane π correspond by the transformation Σ_h of Čech if, and only if, their coordinates are related by the equations

$$\xi_0 = 0, \qquad \sigma\xi_1 = y^1(y^2)^2, \qquad \sigma\xi_2 = (y^1)^2y^2,$$
$$\sigma\xi_3 = -\, y^0y^1y^2 + h[\beta(y^1)^3 + \gamma(y^2)^3].$$

The equation of the polar plane of P with respect to the quadric Q_K is

$$y^2x^1 + y^1x^2 + (k_1y^1 + k_2y^2 - y^0)x^3 = 0.$$

The local plane coordinates of this plane are therefore given by

$$\xi_0 = 0, \qquad \sigma\xi_1 = y^2, \qquad \sigma\xi_2 = y^1, \qquad \sigma\xi_3 = k_1y^1 + k_2y^2 - y^0,$$

where k_1, k_2 are defined by (9.3). These equations may be written in the form

$$(10.1) \qquad \xi_0 = 0, \qquad \sigma\xi_1 = \lambda y^1, \qquad \sigma\xi_2 = y^1, \qquad \sigma\xi_3 = (k_1 + k_2\lambda)y^1 - y^0,$$

where y^2 is replaced by λy^1. The local coordinates of the plane π which corresponds to a point P on t_λ are found to be

$$\xi_0 = 0, \qquad \sigma\xi_1 = \lambda^2(y^1)^3, \qquad \sigma\xi_2 = \lambda(y^1)^3,$$
$$\sigma\xi_3 = -\, \lambda y^0(y^1)^2 + h(\beta + \gamma\lambda^3)(y^1)^3.$$

Multiplying equations (10.1) by $(y^1)^2$ and replacing k_1 and k_2 by their values from (9.3) reveals that the polar plane (10.1) and the plane π coincide if, and only if, h and K are related by the equation

$$(10.2) \qquad hK^2 + (h + 1)K + h = 0.$$

The roots of this equation are reciprocals.

THEOREM 10.1 *The polar plane of an arbitrary point P in π_x with respect to the quadric Q_K determined for the direction of the line joining x and P is the plane* *which corresponds to P in the transformation Σ_h of Čech, where h and K are related by* (10.2). *There are two families of quadrics Q_K and $Q_{K^{-1}}$ which serve to characterize geometrically each particular transformation Σ_h.*

A general pair of quadrics Q_K, $Q_{K^{-1}}$ will be called *Čech associate quadrics.* These quadrics become *self-associates* when $K = \pm 1$. Equation (10.2) has double roots -1 if $h=1$ and double roots $+1$ if $h=-1/3$.

Specializations of Theorem 10.1 yield geometric characterizations for the correspondence of Segre Σ_1, the correspondence of Moutard [6, p. 508] $\Sigma_{-1/3}$, the polarity of Lie Σ_0, and the associate correspondence of Segre [4, p. 233] Σ_{-1}. The correspondence of Segre, the correspondence of Moutard, and the polarity of Lie are induced by the Čech self-associate quadrics Q_{-1} (the quadrics of Davis), the Čech self-associate quadrics Q_1 (the quadrics of the Moutard pencil), and the asymptotic associate quadrics, respectively.

Let $\mu = K\lambda$, $\mu_i = K_i\lambda$, $i=1, 2, 3$, where K_1, K_2 are the roots of (10.2), and $K_3=1$. The cross ratio equation

$$(\mu_1, \mu_2, \mu_3, \mu) = -1$$

is satisfied if, and only if, $K=-1$. The following new geometric characterization of conjugate directions may now be stated.

THEOREM 10.2. *The lines of contact of an arbitrary pair of cones $\mathcal{C}_{\lambda,K}$ and $\mathcal{C}_{\lambda,K^{-1}}$ with the plane π_x separate harmonically the conjugate tangents t_λ and $t_{-\lambda}$.*

11. Pangeodesics. Moutard quadric. Asymptotic osculating quadrics. The equation of the cone $\mathcal{C}_{\lambda,K}$ is found by replacing k_1 and k_2 in equation (3.4) by their values in (9.3) to be

$$\begin{aligned} K^2(x^1)^2 - 2K\lambda^{-1}x^1x^2 + \lambda^{-2}(x^2)^2 + 2\rho a_{13}x^1x^3 \\ + 2\rho a_{23}x^2x^3 + 2\rho a_{33}(x^3)^2 = 0, \end{aligned} \tag{11.1}$$

where $\rho = (K^2+K+1)/2(\beta+\gamma\lambda^3)$ and a_{13}, a_{23}, a_{33} are the coefficients of x^1x^3, x^2x^3, $(x^3)^2$, respectively, in equation (3.4) after k_1 and k_2 have been replaced by their values in (9.3). The necessary and sufficient condition that $\mathcal{C}_{\lambda,K}$ reduce to two planes is that the discriminant of (11.1) be equal to zero. This condition is found to be

$$\begin{aligned} (K+1)k_3 = {} & \lambda^{-2}\alpha(1-K)(\beta+K^2\gamma\lambda^3)\lambda' \\ & - 2\lambda^{-1}\alpha(K+1)^2[\beta^2+(K^3+1)\beta\gamma\lambda^3+K^3\gamma^2\lambda^6] \\ & - 2^{-1}(K+1)\theta_{uv} + \alpha[K\beta_u\lambda^{-1} - (2K^2+3K+3)\beta_v \\ & - (3K^3+3K^2+2K)\gamma_u\lambda^2 + K^2\gamma_v\lambda^3] \\ & - \alpha(K+1)[-\beta(\log\gamma)_u\lambda^{-1} + (K+1)\beta(\log\gamma)_v \\ & + K(K+1)\gamma(\log\beta)_u\lambda^2 - K^2\gamma(\log\beta)_v\lambda^3], \end{aligned} \tag{11.2}$$

where $\alpha = [2\lambda(K^2+K+1)]^{-1}$.

THEOREM 11.1. *The cone* $\mathcal{C}_{\lambda,K}$ *reduces to two planes if, and only if,* k_3 *is defined by the formula* (11.2). *These two planes intersect in a line of* π_x *whose direction is* $K\lambda$.

The following geometric characterizations of the pangeodesics, the Moutard quadric, and asymptotic osculating quadrics can be deduced by substituting proper values of K in formula (11.2).

THEOREM 11.2. *The cone* $\mathcal{C}_{\lambda,K}$ *of a member of the Čech self-associate quadrics* Q_{-1} *reduces to two planes if, and only if, the curve* C_λ *is a pangeodesic.*

THEOREM 11.3. *The cone* $\mathcal{C}_{\lambda,K}$ *of a member of the Čech self-associate quadrics* Q_1 *reduces to two planes if, and only if,* Q_1 *is the Moutard quadric for the direction* λ. *These two planes,* π_x, *and the osculating plane of* C_λ *at* x *form a harmonic set.*

The proof of the second part of the above theorem is left to the reader.

THEOREM 11.4. *The cone* $\mathcal{C}_{\lambda,K}$ *of a member of the asymptotic associate quadrics* Q_0 (Q_∞) *reduces to two planes if, and only if,* Q_0 (Q_∞) *is the asymptotic osculating quadric* Q_u (Q_v) *of Bompiani for the curve* C_λ *at* x.

12. The class $\mathfrak{P}$ of generalized pangeodesics. The equation of the polar plane of any point P on an arbitrary tangent t_μ with respect to the cone $\mathcal{C}_{\lambda,K}$ is found by substituting the values for k_1 and k_2, given by (9.3), in equation (3.5) to be

$$2K\lambda^2x^1 - 2\lambda x^2 + Lx^3 = 0, \tag{12.1}$$

where L is defined by

$$\begin{aligned} L = \{ & [-(K+1)\beta\lambda + 2\beta\mu - 2K^2\gamma\lambda^4 + (K^2+K)\gamma\lambda^3\mu]\lambda' \\ & + (K^2+K+1)^{-1}[(2K^2+2K+1)\beta^2\lambda - K\mu\beta^2 \\ & + (-K^3+2K^2+2K+1)\beta\gamma\lambda^4 + (K^4+2K^3+2K^2-K)\beta\gamma\mu \\ & - K^3\gamma^2\lambda^7 + (K^4+2K^3+2K^2)\gamma^2\lambda^6\mu] - \beta_u\lambda\mu \\ & + (2K^2+3K+3)\beta_v\lambda^3 + (3K^2+3K+2)\gamma_u\lambda^4\mu - K^2\gamma_v\lambda^6 \\ & + K^2\gamma(\log\beta)_u\lambda^5 + (K^2+K+1)\gamma(\log\beta)_u\lambda^4\mu - (K+1)\beta(\log\gamma)_u\lambda^2 \\ & - (K^2+K)\gamma(\log\beta)_v\lambda^5\mu + (K^2+K+1)\beta(\log\gamma)_v\lambda^3 + \beta(\log\gamma)_v\lambda^2\mu \\ & + (K^2+K+1)(\lambda+\mu)(\theta_{uv}+2k_3)\}/(K\lambda-\mu)(\beta+\gamma\lambda^3). \end{aligned} \tag{12.2}$$

The polar plane of an arbitrary point on t_μ with respect to the cone $\mathcal{C}_{\lambda,K}$ will be designated by q_μ. The polar plane q_μ intersects π_x in the line whose direction is $K\lambda$. The equation of the osculating plane of the curve $C_{K\lambda}$ at x is given by (12.1) where

$$L = \lambda_u + K\lambda\lambda_v + \beta/K - \theta_u\lambda + K\theta_v\lambda^2 - K^2\gamma\lambda^3. \tag{12.3}$$

The polar plane q_μ, therefore, coincides with the osculating plane of $C_{K\lambda}$ at x

if, and only if, the values of L defined by (12.2) and (12.3) are equal. When $K=-1$, this condition is independent of the choice of μ and the following theorem can be stated.

THEOREM 12.1. *The polar plane of any point P in the tangent plane of S at x with respect to a cone $\mathcal{C}_{\lambda,K}$ of a member of the Čech self-associate quadrics Q_{-1} coincides with the osculating plane of $C_{K\lambda}$ if, and only if, C_λ is a pangeodesic.*

In this case in which C_λ is a pangeodesic the cone $\mathcal{C}_{\lambda,K}$ reduces to two planes (Theorem 11.2) which separate harmonically the plane π_x and the polar plane of an arbitrary point of π_x with respect to the planes of $\mathcal{C}_{\lambda,K}$.

In case μ is the conjugate direction $-\lambda$ of λ, the equation formed by equating the values of L given by (12.2) and (12.3) reduces to

$$
\begin{aligned}
&[2(K+2)\beta+(3K^2+2K+1)\gamma\lambda^3]\lambda_u+[(K^2+2K+3)\beta \\
&\quad+(4K^2+2K)\gamma\lambda^3]\lambda\lambda_v=(K+2)\beta_u\lambda+(K^2+2K+3)\beta_v\lambda^2 \\
&\quad-(3K^2+2K+1)\gamma_u\lambda^4-(2K^2+K)\gamma_v\lambda^5 \\
&\quad+K^{-1}(K-1)(K+1)^2(K^2+K+1)^{-1}(\beta+\gamma\lambda^3)(\beta+K^3\gamma\lambda^3).
\end{aligned}
\tag{12.4}
$$

DEFINITION. A system of curves each curve C_λ of which is characterized by the property that as x moves along C_λ the polar plane $q_{-\lambda}$ coincides with the osculating plane of $C_{K\lambda}$ at x will be called *a system $\mathfrak{P}_K$ of the class $\mathfrak{P}$ of generalized pangeodesics of S at x.* The differential equation of the system $\mathfrak{P}_K$ is (12.4).

Since (12.4) reduces to the equation for the pangeodesics when $K=\pm 1$, the following theorem results.

THEOREM 12.2. *The systems $\mathfrak{P}_1$ and $\mathfrak{P}_{-1}$ of curves of the class $\mathfrak{P}$ of generalized pangeodesics coincide in the system of pangeodesics of S at x.*

Since (12.4) reduces to the equation of the curves of Darboux for $K=0$ and $K=\infty$, the following theorem results.

THEOREM 12.3. *The curves of the systems $\mathfrak{P}_0$ and $\mathfrak{P}_\infty$ of the class $\mathfrak{P}$ of generalized pangeodesics coincide. The curves, thus characterized, are the curves of Darboux.*

It is noteworthy that the quadrics employed in the above characterizations of the pangeodesics and the curves of Darboux are the Čech self-associate quadrics and the asymptotic associate quadrics, respectively.

The following theorem may be verified by the reader.

THEOREM 12.4. *The polar plane of an arbitrary point on t_λ with respect to the characteristic cone $\mathcal{C}_{\lambda,K}$ of the member of the Čech self-associate quadrics Q_{-1} for which $k_3=\theta_{uv}/2$ coincides with the osculating plane of $C_{-\lambda}$ if, and only if, $C_{\lambda*}$ is a hypergeodesic defined by*

$$\lambda^{*\prime} = [2\theta_u + (\log \gamma)_u]\lambda^* - [2\theta_v + (\log \beta)_v]\lambda^{*2}$$

where λ^* *is the* N_λ*-correspondent* [3, p. 538] *of* λ *defined by*

$$\lambda^* = -\gamma\lambda^4/\beta.$$

The cusp-axis of these hypergeodesics is the canonical line $c'(-1/2)$. *This cusp-axis and the directrix* d' *of Wilczynski separate harmonically the projective normal* n' *of Fubini and the canonical tangent* t'.

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